JOURNAL OF DIFFERENTIAL EQUATIONS 54, 248–273 (1984)

Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators, Periodic Problems

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Received February 8, 1983

1. INTRODUCTION

In this paper, we shall study the existence of strong solutions for the following abstract periodic problem in a real separable Hilbert Space H:

$$\left.\begin{aligned} &\frac{du}{dt}(t) + \partial\varphi^t(u(t)) + B(t, u(t)) \ni f(t), \qquad 0 < t < T, && (1.1)\\ &u(0) = u(T), && (1.2)\end{aligned}\right| \text{(P.P.)}$$

where T is a given positive number, $\partial\varphi^t$ is the subdifferential of a time-dependent lower semicontinuous convex function φ^t from H into $[0, +\infty]$ with $\varphi^t \not\equiv +\infty$, and where $B(t, \cdot)$ is a possibly nonmonotone multivalued nonlinear operator from $D(B(t, \cdot)) \subset H$ into H such that $D(\partial\varphi^t) \subset D(B(t, \cdot))$ for all $t \in [0, T]$.

The case when $B(t, \cdot)$ is identically zero has been investigated by many authors; see e.g., Bénilan and Brézis [2], Nagai [13] and Yamada [20]. As for the case when $B(t, \cdot)$ is not identically zero nor monotone, however, it seems that very little study has been devoted to this kind of problem. On the other hand, the Cauchy problem for (1.1) has been studied by several authors, i.e., Attouch and Damlamian [1], Biroli [3], Koi and Watanabe [9] and the author [14, 15]. The main objective of the present paper is to pursue and develop the method employed in [15] for the periodic problem (P.P.).

This paper is composed of five sections. Section 2 contains some preliminaries. In Section 3 we shall formulate main results in two distinctive types of theorems, roughly speaking, according to whether the external force $f(t)$ can be taken arbitrarily large or not. In Section 4 we shall prove them by much the same method based on a Schauder–Tychonoff-type fixed point theorem as in [15]. In Section 5 it will be exemplified that our results give a

0022-0396/84 $3.00

unified abstract treatment for periodic problems of some nonlinear heat equations with a difference term of monotone operators and Navier–Stokes-type equations in bounded regions with periodically moving boundaries.

2. Preliminaries

Let H be a real separable Hilbert space with the inner product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$, which are often denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively, for the sake of simplicity. Since the present paper is a direct continuation of a previous paper [15], we omit the definitions of the subdifferential operator and the upper semicontinuous multivalued mapping and fundamental facts concerning them (see Section 2 of [15]).

We begin with the definition of strong solutions of (P.P.).

Definition 2.1. A function $u(t) \in C_\pi([0, T]; H) = \{u; u(t)$ is continuous from $[0, T]$ into H and $u(0) = u(T)\}$ is said to be a strong solution of (P.P.) if $u(t)$ is an H-valued absolutely continuous function on $[0, T]$ and belongs to $D(\partial\varphi^t)$ for a.e. $t \in [0, T]$, and if there exist two H-valued measurable functions $g(t)$ and $b(t)$ such that $g(t) \in \partial\varphi^t(u(t))$, $b(t) \in B(t, u(t))$ and

$$du(t)/dt + g(t) + b(t) = f(t) \tag{1.1'}$$

hold for a.e. $t \in [0, T]$.

In order to assure the existence of strong solutions of (P.P.), it would be necessary to assume that φ^t depends on t smoothly in a sense. In fact, we here employ the following condition $(A \cdot \varphi^t)_{p,\beta}$.

$(A.\varphi^t)_{p,\beta}$ The following (i)–(iv) hold.

(i) For each $t \in [0, T]$, φ^t is a lower semicontinuous convex function from H into $[0, +\infty]$ with $\varphi^t \not\equiv +\infty$. Furthermore, for each $t_0 \in [0, T]$ and $x_0 \in D(\varphi^{t_0})$, there exist positive constants m_1, m_2, m_3, δ_0 (independant of t_0 and x_0), an H-valued function $x(t)$ on $I(t_0) := [\max(0, t_0 - \delta_0), \min(t_0 + \delta_0, T)]$ such that

$$|x(t) - x_0|_H \leqslant m_1 |t - t_0| (\varphi^{t_0}(x_0) + m_2)^\beta, \qquad 0 \leqslant \beta \leqslant 1, \tag{2.1}$$

and

$$\varphi^t(x(t)) \leqslant \varphi^{t_0}(x_0) + m_3 |t - t_0| (\varphi^{t_0}(x_0) + m_2) \tag{2.2}$$

hold for all $t \in I(t_0)$.

(ii) $\varphi^0(u) = \varphi^T(u)$ for all $u \in H$.

(iii) There exist positive constants K_0 and p such that

$$K_0 |u|_H^p \leqslant \varphi^t(u), \qquad 1 < p < +\infty, \qquad \text{for all} \quad u \in D(\varphi^t). \tag{2.3}$$

(iv) For each $t \in [0, T]$, $\partial\varphi^t$ is strictly monotone, i.e., $(w_1 - w_2, u_1 - u_2)_H = 0$ with $u_i \in D(\partial\varphi^t)$ and $w_i \in \partial\varphi^t(u_i)$ $(i = 1, 2)$ implies $u_1 = u_2$.

Then, in the case of $B(t, \cdot) \equiv 0$, the following theorem holds. (See Bénilan and Brézis [2], Nagai [13] and Yamada [20].)

THEOREM 2.2. *Let $(A.\varphi^t)_{p,\beta}$ be satisfied and $f(t) \in L^2(0, T; H)$. Then the problem* (P.P.) *with $B(t, \cdot) \equiv 0$ has a unique strong solution $u(t) \in C_\pi([0, T]; H)$ satisfying*

$$du(t)/dt \in L^2(0, T; H), \tag{2.6}$$

$\varphi^t(u(t))$ is absolutely continuous on $[0, T]$ and

$$\varphi^0(u(0)) = \varphi^T(u(T)). \tag{2.7}$$

3. RESULTS

In this section, we shall mention our main results of the present paper. To this end, we first recall the following conditions (A.1) and (A.2), which are introduced in [15].

(A.1) For each $t \in [0, T]$ and $L \in (0, +\infty)$, the set $\{u \in H;\ \varphi^t(u) + |u|_H^2 \leqslant L\}$ is compact in H.

(A.2) The following (i)–(iii) hold:

(i) $B(t, u)$ is a convex subset of H for all $t \in [0, T]$ and $u \in D(\partial\varphi^t)$.

(ii) $B(t, \cdot)$ is *measurable* in the following sense: For each function $u \in C([a, b]; H)$ such that $du(t)/dt \in L^2(0, T; H)$ and there exists a function $g(t) \in L^2(0, T; H)$ with $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0, T]$, there exists an H-valued measurable function $b(t)$ such that $b(t) \in B(t, u(t))$ for a.e. $t \in [0, T]$.

(iii) $B(t, \cdot)$ is *demiclosed* in the following sense: If $u_n \to u$ in $C([0, T]; H)$, $g_n \to g$ weakly in $L^2(0, T; H)$ with $g_n(t) \in \partial\varphi^t(u(t))$, $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0, T]$, and if $b_n \to b$ weakly in $L^2(0, T; H)$ with $b_n(t) \in B(t, u_n(t))$ for a.e. $t \in [0, T]$, then $b(t) \in B(t, u(t))$ holds for a.e. $t \in [0, T]$.

We also assume that $B(t, \cdot)$ is dominated by $\partial\varphi^t(\cdot)$ in a sense. First, we introduce:

(A.3) There exist positive numbers k, α, K_1 and a monotone increasing function $l_1(\cdot)$ on $[0, +\infty]$ such that

(i) $|||B(t, u)|||_H^2 \leqslant k\, |\partial^0\varphi^t(u)|_H^2 + l_1(|u|_H)(\varphi^t(u)+1)^2$, $0 \leqslant k < 1$, for a.e. $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$,

(ii) $\langle -\partial\varphi^t(u) - B(t, u), u\rangle_H + \alpha\varphi^t(u) \leqslant K_1$ for a.e. $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$,

(iii) $\varphi^t(0) \leqslant K_1$ for all $t \in [0, T]$, where $|||B(t, u)|||_H = \operatorname{Sup}\{|b|_H;\ b \in B(t, u)\}$,

$\langle -\partial\varphi^t(u) - B(t, u), u\rangle_H = \operatorname{Sup}\{(-g-b, u)_H;\ g \in \partial\varphi^t(u), b \in B(t, u)\}$ and $\partial^0\varphi^t$ denotes the minimal section of $\partial\varphi^t$.

Then our first main theorem is stated as follows.

THEOREM I. *Let* $(\mathrm{A}.\varphi^t)_{p,\beta}$, (A.1), (A.2) *and* (A.3) *be satisfied, and let* $f(t) \in L^2(0, T; H)$. *Then* (P.P.) *has a strong solution* $u(t) \in C_\pi([0, T]; H)$ *satisfying*

$$du(t)/dt \in L^2(0, T; H), \tag{3.1}$$

$g(t), b(t) \in L^2(0, T; H)$, *where* $g(t)$ *and* $b(t)$ *are the sections of* $\partial\varphi^t(u(t))$ *and* $B(t, u(t))$, *respectively, satisfying* (1.1)′, (3.2)

$\varphi^t(u(t))$ *is absolutely continuous on* $[0, T]$ *and* $\varphi^0(u(0) = \varphi^T(u(T))$. (3.3)

Roughly speaking, the above theorem deals with the situation similar to that of Theorem IV in [15], which assures the existence of global (in time t) solutions of Cauchy problem for (1.1) when initial data and external forces are arbitrarily given in $D(\varphi^0)$ and $L^2(0, T; H)$, respectively. On the other hand, if condition (ii) of (A.3) is absent, then there is a case where for certain initial data and external forces, their corresponding local strong solutions of Cauchy problem for (1.1) blow up in a finite time (see Fujita [5], Tsutsumi [19], Ishii [7] and the author [16]). So, in this case, it would be unikely that (P.P.) has a strong solution for an arbitrary $f(t) \in L^2(0, T; H)$. As an analogue of Theorem V of [15], however, one can expect that (P.P.) has a strong solution if $f(t)$ is sufficiently small in some sense. In fact, we have the following result.

THEOREM II. *Let* $(\mathrm{A}.\varphi^t)_{p,\beta}$ *with* $m_2 = 0$, (A.1), (A.2) *and the following* $(\mathrm{A}.4)_{p,\beta}$ *be satisfied:*

$(\mathrm{A}.4)_{p,\beta}$ *Let the following* (i) *and* (ii) *be satisfied:*

(i) $\varphi^t(0) \equiv 0$ *for all* $t \in [0, T]$,

(ii) *There exist two exponents* $\alpha_1 = \alpha_1(p, \beta)$, $\alpha_2 = \alpha_2(p)$ *and a monotone increasing function* $l_2(\cdot)$ *such that*

$$|||B(t, u)|||_H \leqslant l_2(\varphi^t(u) + |u|_H)\{(\varphi^t(u))^{\alpha_1} |\partial^0\varphi^t(u)|_H + (\varphi^t(u))^{\alpha_2}\}$$
$$\textit{for a.e. } t \in [0, T] \textit{ and all } u \in D(\partial\varphi^t),$$

where $\alpha_1 > (1-\beta_*)\{p_*(1-\beta_*)-1\}$, $\alpha_2 > (p_*-1)/p_*$, $\beta_* = \min(\beta, \frac{1}{2})$ *and* $p_* = \max(p, 2)$. *Then there exists a (sufficiently small) positive number* r *independent of* T *such that if* $\sup_{1 \leqslant t \leqslant T} \int_{t-1}^{t} |f(s)|_H^2 \, ds \leqslant r$, *then* (P.P.) *has a strong solution* $u \in C_\pi([0, T]; H)$ *satisfying* (3.1)–(3.3).

Remark 3.1. If one allows that r may depend on T in Theorem II, then the assertion of the theorem holds true under more relaxed conditions that $(\mathrm{A}.4)_{p,\beta}$, i.e., concerning the exponents α_1 and α_2, one has only to assume that $\alpha_1 > (1-\beta_*)(1-2\beta_*)$ and $\alpha_2 > \frac{1}{2}$, which is denoted by $(\mathrm{A}.4)'_{p,\beta}$. In particular, as for the case $0 < T < 1$, Theorem II holds good with $(\mathrm{A}.4)_{p,\beta}$ and $\sup_{1 \leqslant t \leqslant T} \int_{t-1}^{t} |f(s)|_H^2 \, ds$ replaced by $(\mathrm{A}.4)'_{p,\beta}$ and $\int_0^T |f(t)|_H^2 \, dt$, respectively. (For a proof see Section 4.)

Remark 3.2. Even if one removes assumption (iv) of $(\mathrm{A}.\varphi^t)_{p,\beta}$ in Theorems I and II, the assertions of the theorems are still valid (see Section 4).

4. Proofs of Theorems

4.1. *Some Lemmas*

In this subsection, we prepare some results on the following auxiliary equation:

$$\left.\begin{aligned} &du_h(t)/dt + \partial\varphi^t(u_h(t)) \ni -h(t) + f(t), \qquad 0 < t < T, &(4.1)\\ &u_h(0) = u_h(T). &(4.2)\end{aligned}\right| \ (\mathrm{P.P.})^*$$

Here T is a given positive number and $f(t)$ is a fixed element in $L^2(0, T; H)$. Under assumption $(\mathrm{A}.\varphi^t)_{p,\beta}$, Theorem 2.2 assures that for all $h(t) \in L^2(0, T; H)$, there exists a unique strong solution $u_h(t)$ of $(\mathrm{P.P.})^*$ in $C_\pi([0, T]; H)$. Therefore, for each T and $f \in L^2(0, T; H)$, we can well define an operator $\mathbf{E}_{\pi,f,T}$ from $L^2(0, T; H)$ into $C_\pi([0, T]; H)$ by

$$\mathbf{E}_{\pi,f,T} : h \mapsto u_h.$$

In what follows, we often write $\mathbf{E}$ instead of $\mathbf{E}_{\pi,f,T}$ for the sake of simplicity. On the continuity of $\mathbf{E}$, we get (cf. Lemma 3.10 in [15]):

Lemma 4.1. *Let* $(\mathrm{A}.\varphi^t)_{p,\beta}$ *and* (A.1) *be satisfied. Let* $\{h^n\}$ *be a sequence in* $L^2(0, T; H)$ *such that* h^n *converges to* h *weakly in* $L^2(0, T; H)$ *as* $n \to +\infty$. *Then* $\mathbf{E}_{\pi,f,T}(h^n)$ *converges to* $\mathbf{E}_{\pi,f,T}(h)$ *strongly in* $C_\pi([0, T]; H)$ *as* $n \to +\infty$.

To prove this lemma, we prepare the following lemma.

LEMMA 4.2. *Let $f(t) \in L^1(0, T)$ and $e(t)$ be a positive absolutely continuous function on $[0, T]$ such that $e(0) = e(T)$. Suppose that there exist positive constants α, γ and C such that*

$$de(t)^2/dt + \alpha e(t)^{1+\gamma} \leqslant C + |f(t)|\, e(t) \qquad \text{for a.e.} \quad t \in [0, T]. \tag{4.3}$$

Then we have

$$\max_{0 \leqslant t \leqslant T} e(t) \leqslant 2(\sqrt{CT} + |f|_{L^1(0,T)}) + \{(2\,|f|_{L^1(0,T)} + \sqrt{CT})/\alpha T\}^{1/\gamma}. \tag{4.4}$$

Proof. Put $m = \min\{e(t); 0 \leqslant t \leqslant T\}$ and $M = \max\{e(t); 0 \leqslant t \leqslant T\}$. Then, by (4.3), we get

$$M^2 \leqslant m^2 + CT + M\,|f|_{L^1(0,T)},$$

whence follows

$$M \leqslant m + \sqrt{CT} + |f|_{L^1(0,T)}. \tag{4.5}$$

On the other hand, integrating (4.3) over $[0, T]$ and using (4.5), we have

$$\begin{aligned} \alpha T m^{1+\gamma} &\leqslant \int_0^T \alpha e(t)^{1+\gamma}\, dt \\ &\leqslant CT + M\,|f|_{L^1(0,T)} \\ &\leqslant CT + |f|_{L^1(0,T)}\,(m + \sqrt{CT} + |f|_{L^1(0,T)}), \end{aligned}$$

which implies

$$m \leqslant \sqrt{CT} + |f|_{L^1(0,T)} + \{(2\,|f|_{L^1(0,T)} + \sqrt{CT})/\alpha T\}^{1/\gamma}.$$

Then this estimate and (4.5) give (4.4). Q.E.D.

Proof of Lemma 4.1. By virtue of Theorem 2.2, there exists an H-valued absolutely continuous function $v(t)$ on $[0, T]$ such that

$$\left| \begin{aligned} & dv(t)/dt + \partial\varphi^t(v(t)) \ni 0 \qquad \text{for a.e.} \quad t \in [0, T], \\ & v(0) = v(T), \end{aligned} \right. \tag{4.6}$$

and

$$r_0 := \max\{|v(t)|_H + \varphi^t(v(t)); 0 \leqslant t \leqslant T\} + |dv(t)/dt|_{L^2(0,T;H)} < +\infty. \tag{4.7}$$

Let $u^n = \mathbf{E}_{\pi,f,T}(h^n)$. Then miltiplication of (4.1) by $u^n(t) - v(t)$ gives

$$\frac{1}{2}\frac{d}{dt}|u^n(t)-v(t)|^2 + \varphi^t(u^n(t))$$
$$\leqslant \varphi^t(v(t)) + |u^n(t)-v(t)| \left(\left| \frac{dv}{dt}(t) \right| + |h^n(t)| + |f(t)| \right)$$
$$\text{for a.e. } \quad t \in [0, T]. \tag{4.8}$$

Hence, in view of (iii) of $(\mathrm{A}.\varphi^t)_{p,\beta}$ and Lemma 4.2, we find

$$\max_{0 \leqslant t \leqslant T} |u^n(t)| + \int_0^T \varphi^t(u^n(t))\, dt \leqslant C_0 \qquad \text{for all } n, \tag{4.9}$$

where C_0 denotes a general constant depending only on p, K_0, r_0, $|f|_{L^1(0,T;H)}$ and $\sup_n |h^n|_{L^1(0,T;H)}$.

In particular, (4.9) implies that there exists a number $t_0 \in [0, T]$ such that

$$\varphi^{t_0}(u^n(t_0)) \leqslant C_0. \tag{4.10}$$

On the other hand, multiplying (4.1) by $g^n(t) = -du^n(t)/dt - h^n(t) + f(t) \in \partial\varphi^t(u^n(t))$, we obtain, by Proposition 3.4 of [15],

$$|g^n(t)|^2 + \frac{d}{dt}\varphi^t(u^n(t))$$
$$\leqslant |g^n(t)| \{|h^n(t)| + |f(t)|\} + m_1 |g^n(t)| (\varphi^t(u^n(t)) + m_2)^\beta$$
$$+ m_3(\varphi^t(u^n(t)) + m_2)$$
$$\leqslant |g^n(t)|^2/2 + (|h^n(t)| + |f(t)|)^2$$
$$+ (m_1^2 + m_3)(\varphi^t(u^n(t)) + m_2 + 1)^2 \qquad \text{for a.e. } \quad t \in [0, T]. \tag{4.11}$$

Hence, by Gronwall's inequality, (4.9) and (4.10), we deduce

$$\int_0^T |g^n(t)|^2\, dt + \max_{0 \leqslant t \leqslant T} \varphi^t(u^n(t)) \leqslant C_1 \qquad \text{for all } n, \tag{4.12}$$

where C_1 is a constant depending only on C_0 and $\sup_n |h^n|_{L^2(0,T;H)}$. Then we can complete the proof as in the proof of Lemma 3.10 of [15] by using a priori estimates (4.9) and (4.12). Q.E.D.

For each bounded closed convex set K in $L^2(0, T; H)$ endowed with the weak topology of $L^2(0, T; H)$, we introduce a possibly multivalued operator $\mathbf{B}_{\pi,f,T,K}$, which will be simply denoted by $\mathbf{B}_K$ or $\mathbf{B}$, from K into itself by

$$\mathbf{B}_{\pi,f,T,K}(h) = \{b \in K; b(t) \in B(t, \mathbf{E}_{\pi,f,T}(h)(t)) \text{ for a.e. } t \in [0, T]\} \tag{4.13}$$

with domain

$$D(\mathbf{B}_{\pi,f,T,K}) = \{h \in K; \mathbf{B}_{\pi,f,T,K}(h) \neq \phi\}. \tag{4.14}$$

Then, since the continuity of $\mathbf{E}_{\pi,f,T}$ is already known, repeating the very same reasoning as in the proof of Lemma 3.11 of [15], we can prove the following lemma.

LEMMA 4.3. *Let* $(\mathrm{A}.\varphi^t)_{p,\beta}$, (A.1) *and* (A.2) *be satisfied. Then* $G(\mathbf{B}_K)$, *the graph of* $\mathbf{B}_K$, *is closed in* $K \times K$. *Moreover, for each* $h \in D(\mathbf{B}_K)$, $\mathbf{B}_K(h)$ *is a closed convex subset of* K.

4.2. *Proof of Theorem* I

First of all, we prepare the following proposition.

PROPOSITION 4.4. *Let* $(\mathrm{A}.\varphi^t)_{p,\beta}$ *with* $p > 2$ *and* $0 \leqslant \beta \leqslant \frac{1}{2}$, (A.1) *and* (A.2) *be satisfied. Suppose that there exist positive numbers,* k, δ *and* L_1 *such that*

$$|||B(t,u)|||_H^2 \leqslant k\,|\partial^0\varphi^t(u)|_H^2 + L_1\{(\varphi^t(u))^\delta + 1\},$$
$$0 \leqslant k < 1, \qquad 0 \leqslant \delta \leqslant 1, \qquad \text{for all} \quad u \in D(\partial\varphi^t). \tag{4.15}$$

Then, for all $f(t) \in L^2(0,T;H)$, (P.P.) *has a strong solution* $u(t) \in C_\pi([0,T];H)$ *satisfying* (3.1)–(3.3).

Proof. For the sake of the latter calculation, we put

$$C_1 = (6/pK_0)^{q/p}, \qquad 1/p + 1/q = 1,$$
$$C_2 = 2\{(r_0 + 3r_0^p)\,T + (C_1 + 1)\,T^{1-q/2}(r_0^q + |f|^q_{L^2(0,T;H)})\},$$

where r_0 is the constant given in (4.7),

$$C_3 = 2(C_1 + 1)\,T^{1-q/2} + 1,$$
$$\varepsilon_k = (1-k)/2(1+3k),$$
$$C_4 = \{1 + (m_1^2 + m_3)\,C_3\}/2\varepsilon_k,$$

and fix a positive number R such that

$$R^q \geqslant C_2 + (m_1^2 + m_3)(m_2 + 1)\,T + |f|^2_{L^2(0,T;H)}, \tag{4.16}$$
$$R^{2-q} \geqslant 8C_4 k/(1-k)(1-2\varepsilon_k), \tag{4.17}$$
$$R^{2-\delta q} \geqslant 4L_1 C_3^\delta T^{1-\delta}/(1-k), \tag{4.18}$$
$$R^2 \geqslant 4L_1/(1-k). \tag{4.19}$$

Let $h \in K := \{u \in L^2(0, T; H);\ |u|_{L^2(0,T;H)} \leqslant R\}$ and $u = \mathbf{E}_{\pi,f,T}(h)$. Then, since (4.8) hold with u^n and h^n replaced by u and h, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t)-v(t)|^2 &+ \varphi^t(u(t))\\ &\leqslant r_0 + 3r_0^p/p + K_0\,|u(t)|^p/2\\ &\quad + (C_1+1)(|dv(t)/dt|^q + |h(t)|^q + |f(t)|^q)/q.\end{aligned}$$

Hence, by (iii) of $(\mathrm{A}.\varphi^t)_{p,\beta}$ and (4.16), we obtain

$$\int_0^T \varphi^t(u(t))\,dt \leqslant C_2 + 2(C_1+1)\,T^{1-q/2}R^q \leqslant C_3R^q. \tag{4.20}$$

Furthermore, by the same verification as for (4.11), we see

$$\begin{aligned}|g(t)|^2 &+ \frac{d}{dt}\varphi^t(u(t))\\ &\leqslant |g(t)|^2/2 + |h(t)|^2/2 + \varepsilon_k\,|g(t)|^2\\ &\quad + \{|f(t)|^2 + (m_1^2+m_3)(\varphi^t(u(t)) + m_2 + 1)\}/2\,\varepsilon_k\\ &\quad \text{for a.e.}\quad t \in [0, T],\end{aligned} \tag{4.21}$$

where $g(t) = -du(t)/dt - h(t) + f(t) \in \partial\varphi^t(u(t))$.

Then, integration of (4.21) over $[0, T]$, together with (4.16) and (4.20), gives

$$\begin{aligned}(1/2-\varepsilon_k)\,&|g(t)|^2_{L^2(0,T;H)}\\ &\leqslant R^2/2 + \{|f|^2_{L^2(0,T;H)} + (m_1^2+m_3)(m_2+1)\,T + (m_1^2+m_3)\,C_3R^q\}/2\varepsilon_k\\ &\leqslant R^2/2 + C_4R^q.\end{aligned} \tag{4.22}$$

Let $b(t)$ be an arbitrary H-valued measurable function such that $b(t) \in B(t, u(t)) = B(t, \mathbf{E}(h)(t))$ for a.e. $t \in [0, T]$. Then it follows from (4.15), (4.20), (4.22) and (4.17)–(4.19) that

$$\begin{aligned}|b|^2_{L^2(0,T;H)} &\leqslant (kR^2 + 2kC_4R^q)/(1-2\varepsilon_k) + L_1\{(C_3R^q)^\delta\,T^{1-\delta} + 1\}\\ &\leqslant (1+k)\,R^2/2 + (1-k)\,R^2/2 \leqslant R^2.\end{aligned}$$

Consequently, from (ii) of (A.2) and Lemma 4.3, we see that $D(\mathbf{B}_K) = K$; $G(\mathbf{B}_K)$ is closed in $K \times K$; and that $\mathbf{B}_K(h)$ is a nonempty closed convex subset of K for all $h \in K$. Then Proposition 2.6 of [15] assures that $\mathbf{B}_{\pi,f,T,K}$ is an upper semicontinuous mapping from K into itself. Hence, by virtue of

Theorem 2.7 of [15], $\mathbf{B}_{\pi,f,T,K}$ has a fixed point $\bar{b}$ in K; i.e., $\bar{b} \in \mathbf{B}_{\pi,f,T,K}(\bar{b})$. In other words, $\bar{u} = \mathbf{E}_{\pi,f,T}(\bar{b})$ satisfies

$$\left| \begin{array}{l} d\bar{u}(t)/dt + \partial\varphi^t(\bar{u}(t)) + \bar{b}(t) \ni f(t) \qquad \text{for a.e.} \quad t \in [0, T], \\ \bar{b}(t) \in B(t, \bar{u}(t)) \qquad \text{for a.e.} \quad t \in [0, T], \\ \bar{u}(0) = \bar{u}(T). \end{array} \right.$$

Thus, we have shown that $\bar{u}$ is the desired strong solution of (P.P.) (for relation (3.3), see Remark 3.3 of [15]). Q.E.D.

We are going to prove Theorem I with the aid of the following approximate equations:

$$\left.\begin{array}{ll} du_\varepsilon(t)/dt + \partial\varphi^t_{\varepsilon,r}(u_\varepsilon(t)) + B(t, u_\varepsilon(t)) \ni f(t), 0 < t < T, & \quad (4.23) \\ u_\varepsilon(0) = u_\varepsilon(T), & \quad (4.24) \end{array}\right| \text{(P.P.)}_{\varepsilon,r}$$

where we put

$$\begin{aligned} \varphi^t_{\varepsilon,r}(u) &= \varphi^t_\varepsilon(u) + I_r(u), \\ \varphi^t_\varepsilon(u) &= \varphi^t(u) + \varepsilon(\varphi^t(u))^3/3, \qquad 0 < \varepsilon \leqslant 1, \\ I_r(u) &= 0 \qquad \text{if} \quad |u|_H \leqslant r, \\ &= +\infty \qquad \text{if} \quad |u|_H > r, \quad r > 0. \end{aligned}$$

Under the same assumptions as in Theorem I, the existence of strong solutions $u_\varepsilon(t)$ of $\text{(P.P.)}_{\varepsilon,r}$ is assured by Proposition 4.4 and the following Lemma. (For a proof of this proposition, see the Appendix.)

LEMMA 4.5. *Let* $(\text{A}.\varphi^t)_{p,\beta}$ *and* (iii) *of* (A.3) *be satisfied. Then we have*

(i) $\varphi^t_{\varepsilon,r}$ *satisfies* $(\text{A}.\varphi^t_{\varepsilon,r})_{p,\beta}$ *with* $0 \leqslant \beta < \frac{1}{3}$ *and* $3 < p$,

(ii) $D(\partial\varphi^t_{\varepsilon,r}) = D(\partial\varphi^t_\varepsilon) \cap D(\partial I_r)$ *and* $\partial\varphi^t_{\varepsilon,r}(\cdot) = \partial\varphi^t_\varepsilon(\cdot) + \partial I_r(\cdot)$,

(iii) $D(\varphi^t_\varepsilon) = D(\varphi^t)$, $D(\partial\varphi^t_\varepsilon) = D(\partial\varphi^t)$ *and* $\partial\varphi^t_\varepsilon(\cdot) = (1 + \varepsilon(\varphi^t(\cdot))^2)\,\partial\varphi^t(\cdot)$.

Proof of Theorem I. It is obvious by the above lemma that all assumptions in Proposition 4.4 are satisfied with φ^t replaced by $\varphi^t_{\varepsilon,r}$. Then, for all $\varepsilon > 0$ and $r > 0$, $\text{(P.P.)}_{\varepsilon,r}$ has strong solutions u_ε satisfying (3.1)–(3.3). We are going to show below that u_ε converges to the desired strong solution of (P.P.) as $\varepsilon \downarrow 0$ for a sufficiently large r.

Multiplying (4.23) by $u_\varepsilon(t)$ and using (ii) of (A.3) and (ii)–(iii) of Lemma 4.5, we get

$$\frac{1}{2}\frac{d}{dt}|u_\varepsilon(t)|^2 + \alpha'\varphi^t_\varepsilon(u_\varepsilon(t))$$
$$\leqslant K_1 + \varepsilon K_1^3 + |f(t)|\,|u_\varepsilon(t)| \qquad \text{for a.e.} \quad t \in [0, T], \tag{4.25}$$

where $\alpha' = \min(\alpha, 1)$. Then, since $K_0|u|^p \leqslant \varphi^t(u) \leqslant \varphi^t_\varepsilon(u)$, Lemma 4.2 gives

$$\max_{0 \leqslant t \leqslant T} |u_\varepsilon(t)| \leqslant C_0 \qquad \text{for all} \quad \varepsilon \in (0, 1],$$

where C_0 denotes a general constant depending on α, p, K_0, K_1, T and $|f|_{L^1(0,T;H)}$ but not on r.

Therefore, by taking r sufficiently large, we can forget the parameter r, in other words, $\partial I_r(u_\varepsilon(t)) \equiv 0$, i.e., $\partial\varphi^t_{\varepsilon,r}(u_\varepsilon(t)) \equiv \partial\varphi^t_\varepsilon(u_\varepsilon(t))$ for all $\varepsilon \in (0, 1]$ and $t \in [0, T]$. Moreover, integration of (4.25) on $[0, T]$ yields

$$\int_0^T \varphi^t(u_\varepsilon(t))\,dt \leqslant C_0 \qquad \text{for all} \quad \varepsilon \in (0, 1]. \tag{4.26}$$

On the other hand, multiplying (4.23) by $g_\varepsilon(t) = -du_\varepsilon(t)/dt - b_\varepsilon(t) + f(t) \in \partial\varphi^t_\varepsilon(u_\varepsilon(t))$ with $b_\varepsilon(t) \in B(t, u_\varepsilon(t))$, we obtain, by (i) of (A.3),

$$
\begin{aligned}
&|g_\varepsilon(t)|^2 + \frac{d}{dt}\varphi^t_\varepsilon(u_\varepsilon(t)) \\
&\quad\leqslant |g_\varepsilon(t)|^2/2 + |b_\varepsilon(t)|^2/2 + \varepsilon_k|g_\varepsilon(t)|^2 + m_3(\varphi^t_\varepsilon(u_\varepsilon(t)) + m_2) \\
&\qquad + \{|f(t)|^2 + m_1^2(\varphi^t_\varepsilon(u_\varepsilon(t)) + m_2)^{2\beta}\}/2\varepsilon_k \\
&\quad\leqslant (1 - 3k\varepsilon_k)|g_\varepsilon(t)|^2 + (m_3 + m_1^2/2\varepsilon_k + l_1(C_0)/2)(\varphi^t_\varepsilon(u_\varepsilon(t)) + m_2 + 1)^2 \\
&\qquad + |f(t)|^2/2\varepsilon_k \qquad \text{for all} \quad \varepsilon \in (0, 1] \ \text{and a.e.} \ t \in [0, T],
\end{aligned} \tag{4.27}
$$

where $\varepsilon_k = (1-k)/2(1+3k)$ (see Proposition 3.4. of [15], and (a.1)–(a.2) in the Appendix).

Then it easily follows from (4.26) and Gronwall's inequality that

$$\max_{0 \leqslant t \leqslant T} \varphi^t_\varepsilon(u_\varepsilon(t)) \leqslant C_1 \qquad \text{for all} \quad \varepsilon \in (0, 1], \tag{4.28}$$

where C_1 denotes a general constant depending on C_0, m_1, m_2, m_3, k and $|f|_{L^2(0,T;H)}$. Then, integrating (4.27) over $[0, T]$, we have, by (4.26) and (4.28),

$$|g_\varepsilon(t)|_{L^2(0,T;H)} \leqslant C_1 \qquad \text{for all} \quad \varepsilon \in (0, 1],$$

whence we also find that $du_\varepsilon(t)/dt$ and $b_\varepsilon(t)$ are bounded in $L^2(0, T; H)$. Hence, since $\{u_\varepsilon(t)\}_\varepsilon$ is equicontinuous on $[0, T]$ and forms a relatively compact set in H for all $t \in [0, T]$, Ascoli's theorem assures that there exists a sequence $\{\varepsilon_n\}$ tending to zero as $n \to +\infty$ such that

$$\begin{array}{lll} u_{\varepsilon_n}(t) \to u(t) & \text{strongly in} & C_\pi([0, T]; H), \\ du_{\varepsilon_n}(t)/dt \to du(t)/dt & \text{weakly in} & L^2(0, T; H), \\ g_{\varepsilon_n}(t) \to g(t) & \text{weakly in} & L^2(0, T; H). \\ b_{\varepsilon_n}(t) \to b(t) & \text{weakly in} & L^2(0, T; H). \end{array}$$

Hence, in view of (4.28) and (iii) of Lemma 4.5, we see that $g_{\varepsilon_n}(t)/(1 + \varepsilon_n(\varphi^t(u_{\varepsilon_n}(t)))^2 \in \partial\varphi^t(u_{\varepsilon_n}(t))$ converges to $g(t)$ weakly in $L^2(0, T; H)$.

Then, by virtue of Proposition 1.1 of Kenmochi [8], we know $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0, T]$. Hence, by (iii) of (A.2), $b(t) \in B(t, u(t))$ for a.e. $t \in [0, T]$. Thus $u(t)$ is proved to be the desired strong solution of (P.P.).

Q.E.D.

4.3. *Proof of Theorem* II

In order to derive favorable a priori estimates, we need the following lemma, which is parallel with, but more delicate than, Lemma 4.2.

LEMMA 4.6. *Let $f(t) \in L^1(0, T)$ and $e(t)$ be a positive absolutely continuous function on $[0, T]$, $T \geqslant 1$, such that $e(0) = e(T)$ and*

$$de(t)/dt + \alpha e(t)^{p-1} \leqslant C\,|f(t)| \qquad \text{for a.e.} \quad t \in [0, T], \tag{4.29}$$

where α and C are positive constants and $p > 1$.

Let $\sup_{t \in [r_0, T]} \int_{t-r_0}^{t} |f(s)|\,ds \leqslant R r_0$ with $r_0 = \min(T, R^{(2-p_*)/(p_*-1)})$ and $p_* = \max(p, 2)$. Then there exists a monotone increasing function C_1 of R depending only on α, C and p but not on T such that

$$\max_{0 \leqslant t \leqslant T} e(t) \leqslant C_1 R^{1/(p_*-1)}.$$

Proof. Integration of (4.29) on $[0, T]$ gives

$$\alpha \int_0^T e(t)^{p-1}\,dt \leqslant C \int_0^T |f(t)|\,dt \leqslant C R r_0 (T/r_0 + 1) \leqslant 2CRT.$$

Then there exists a $t_0 \in [0, T]$ such that

$$e(t_0) \leqslant (2CR/\alpha)^{1/(p-1)}. \tag{4.30}$$

Here, by virtue of the periodicity of $e(t)$, we may assume $t_0 = 0$ without loss of generality. Hence, for the case $p \geqslant 2$, we can prove the assertion by essentially the same reasoning as in Lemma 4.3 of [15]. As for the case $1 < p < 2$, we put

$$\phi_a(t) = ([e(a)^{2-p} - \alpha(2-p)(t-a)]^+)^{1/(2-p)} + C\int_a^t |f(s)|\, ds,$$

$a \in [0, T)$, where $[r]^+ = \max(r, 0)$. Then $\phi_a(t)$ satisfies

$$\left| \begin{array}{ll} d\phi_a(t)/dt + \alpha\phi_a(t)^{p-1} \geqslant C\,|f(t)| & \text{for a.e.} \quad t \in [a, T] \\ \phi_a(a) = e(a). & \end{array} \right.$$

Consequently, we find

$$e(t) \leqslant \phi_a(t) \qquad \text{for all} \quad t \in [a, T] \tag{4.31}$$

Now, we claim that C_1 can be taken as follows:

$$C_1 = 2((2C/\alpha)^{1/(p-1)} R^{(2-p)/(p-1)} + C) + (2C/\alpha(2-p))^{1/p-1} R^{(2-p)/(p-1)}. \tag{4.32}$$

Suppose that this is not true. Then, since $e(0) < C_1 R$ by (4.30), there exists a $t_1 \in (0, T]$ such that $e(t_1)$ attains $C_1 R$ for the first time. Hence, we have, by (4.31),

$$\begin{aligned} C_1 R &\leqslant \phi_0(t_1) \leqslant e(0) + C\int_0^{t_1} |f(s)|\, ds \\ &= (2C/\alpha)^{1/(p-1)} R^{(2-p)/(p-1)} R + C(\tau_0 + 1) R \qquad \text{if} \quad t_1 \in [0, \tau_0], \end{aligned} \tag{4.33}$$

and

$$C_1 R \leqslant \phi_{t_1 - \tau_0}(t_1) \leqslant C(\tau_0 + 1) R \qquad \text{if} \quad t_1 \in [\tau_0, T], \tag{4.34}$$

since $[e(t_1 - \tau_0)^{2-p} - \alpha(2-p)\,\tau_0]^+ = 0$, where $\tau_0 = (C_1 R)^{2-p}/\alpha(2-p)$. By simple calculations, however, it is shown that (4.33) or (4.34) contradicts (4.32), the definition of C_1. Q.E.D.

Proof of Theorem II. First of all, we fix a (sufficiently small) positive number R satisfying the following conditions.

$$l_2(C_1 + C_3)\, C_3^{\alpha_1} C_4^{1/2} R^{[\alpha_1 - (1-\beta)\{p(1-\beta)-1\}]/(p-1)(1-\beta)} < \tfrac{1}{2}, \tag{4.35}$$

$$l_2(C_1 + C_3)\, C_2^{\alpha_2} R^{q\alpha_2 - 1} < \tfrac{1}{2}, \tag{4.36}$$

$$2l_2(C_1 + C_3)^2\, C_3^{2\alpha_1} C_4 R^{2[\alpha_1 - (1/2-\beta)\{p(1-\beta)-1\}]/(p-1)(1-\beta)} < \tfrac{1}{2}, \tag{4.37}$$

$$2l_2(C_1 + C_3)^2\, C_3^{2\alpha_2 - 1} C_2 R^{2(\alpha_2 - \beta)/(p-1)(1-\beta)} < \tfrac{1}{2}, \tag{4.38}$$

$$C_3 R^{1/(p-1)(1-\beta)} \leqslant 1, \tag{4.39}$$

where C_1 is the monotone increasing function of R given in Lemma 4.6, and

$$C_2 = 2C_1 + C_1^2/2, \qquad q = p/(p-1),$$
$$C_3 = 3 + C_2(1 + m_3)\, R^{(1-2\beta)/(p-1)(1-\beta)} + 2m_1^2 C_2^{2\beta},$$
$$C_4 = 4\{(C_3 + 3) + m_3 C_2 + 2m_1^2 C_2^{2\beta}\}.$$

Here and henceforth, we use simple notations p, β and α_2 instead of p_*, β_* and $\alpha_2^* = \min(\alpha_2, 1)$, respectively, if no confusion arises. We prove the theorem in two distinctive cases.

I. *The case* $r_0 = R^{(2-p)/(p-1)} \leqslant T$. Let

$$h, f \in K := \Big\{ v \in L^2(0, T; H);$$
$$\sup \Big\{ \int_{t-r_0}^{t} |v(s)|\, ds;\, t \in [r_0, T] \Big\} \leqslant R^{1/(p-1)},$$
$$\sup \Big\{ \int_{t-r_0}^{t} |v(s)|^2\, ds;\, t \in [r_0, T] \Big\} \leqslant R^{1/(p-1)(1-\beta)} \Big\},$$

and $u \in \mathbf{E}_{\pi, f, T}(h)$. Then, since $\varphi^t(0) \equiv 0$, multiplication of (4.1) by $u(t)$ yields

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \varphi^t(u(t)) \leqslant (|f(t)| + |h(t)|)\,|u(t)| \qquad \text{for a.e.} \quad t \in [0, T], \quad (4.40)$$

whence follows, by (iii) of $(\mathrm{A}.\varphi^t)_{p,\beta}$,

$$d\,|u(t)|/dt + K_0\,|u(t)|^{p-1} \leqslant |f(t)| + |h(t)| \qquad \text{for a.e.} \quad t \in [0, T].$$

Then Lemma 4.6 assures that there exists a monotone increasing function C_1 of R independent of T such that

$$\max_{0 \leqslant t \leqslant T} |u(t)| \leqslant C_1 R^{1/(p-1)}. \qquad (4.41)$$

Hence, by integrating (4.40) on $[t - r_0, t]$, we get

$$\sup_{t \in [r_0, T]} \int_{t-r_0}^{t} \varphi^s(u(s))\, ds \leqslant C_2 R^{2/(p-1)}. \qquad (4.42)$$

Furthermore, we claim

$$\max_{0 \leqslant t \leqslant T} \varphi^t(u(t)) \leqslant C_3 R^{1/(p-1)(1-\beta)}. \qquad (4.43)$$

Suppose that this does not hold. Then, since (4.42) implies that there exists a time $t_0 \in [0, T]$ such that

$$\varphi^{t_0}(u(t_0)) \leqslant C_2 R^{2/(p-1)} < C_3 R^{1/(p-1)(1-\beta)}, \tag{4.44}$$

there exists a time $t_1 \in [t_0, t_0 + T]$ such that $\varphi(t_1)$ attains $C_3 R^{1/(p-1)(1-\beta)}$ for the first time in $[t_0, t_0 + T]$, where we put $\varphi(t) = \varphi^t(u(t))$ for $t \in [t_0, T]$ and $\varphi(t) = \varphi^{t-T}(u(t-T))$ for $t \in [T, t_0 + T]$. Let us here recall the relation (see (4.11)),

$$\frac{1}{4}|g(t)|^2 + \frac{d}{dt}\varphi^t(u(t)) \leqslant |h(t)|^2/2 + 2|f(t)|^2 + 2m_1^2(\varphi^t(u(t)))^{2\beta}$$
$$+ m_3\varphi^t(u(t)) \qquad \text{for a.e.} \quad t \in [0, T], \tag{4.45}$$

where $g(t) = -du(t)/dt - h(t) + f(t) \in \partial\varphi^t(u(t))$.

Then, integrating (4.45) on $[s, t_1]$, we get, by (4.42) and (4.39),

$$\varphi(t_1) \leqslant \varphi(s) + 5R^{1/(p-1)(1-\beta)}/2 + 2m_1^2(C_2 R^{2/(p-1)})^{2\beta}|t-s|^{1-2\beta}$$
$$+ m_3 C_2 R^{2/(p-1)} \qquad \text{for all} \quad s \in [\max(t_0, t_1 - r_1), t_1], \tag{4.46}$$

where $r_1 = R^{(1-2\beta)/(p-1)(1-\beta)}$. Putting $s = t_0$ in (4.46) for the case $t_1 - r_1 \leqslant t_0$, or integrating (4.46) with respect to s over $[t_1 - r_1, t_1]$ for the case $t_1 - r_1 \geqslant t_0$, we deduce, from (4.42) and (4.44),

$$C_3 R^{1/(p-1)(1-\beta)} < [C_2(1 + m_3) R^{(1-2\beta)/(p-1)(1-\beta)} + 3 + 2m_1^2 C_2^{2\beta}]$$
$$\times R^{1/(p-1)(1-\beta)},$$

which contradicts the definition of C_3. Thus (4.43) is verified. Moreover, integration of (4.45) on $[t - r_0, t]$, together with (4.42) and (4.43), gives

$$\sup_{t \in [r_0, T]} \int_{t-r_0}^{t} |g(s)|^2\, ds \leqslant C_4 R^{\{2+p(2\beta-1)\}/(p-1)}, \tag{4.47}$$

Then, for all H-valued measurable functions $b(t)$ with $b(t) \in B(t, u(t)) \equiv B(t, \mathbf{E}(h)(t))$, in view of $(\mathrm{A}.4)_{p,\beta}$, (4.41), (4.42), (4.43), (4.47) and (4.35)–(4.39), we get

$$\sup_{t \in [r_0, T]} \int_{t-r_0}^{t} |b(s)|\, ds$$
$$\leqslant l_2(C_1 + C_3)\{C_3^{\alpha_1^*} R^{\alpha_1/(p-1)(1-\beta)} C_4^{1/2} R^{\{2+p(2\beta-1)\}/2(p-1)} R^{(2-p)/2(p-1)}$$
$$+ C_2^{\alpha_2^*} R^{2\alpha_2^*/(p-1)} R^{(2-p)(1-\alpha_2^*)/(p-1)}\}$$
$$< R^{1/(p-1)},$$

$$\begin{aligned}
&\sup_{t\in[r_0,T]}\int_{t-r_0}^{t}|b(s)|^2\,ds\\
&\quad\leqslant 2l_2(C_1+C_3)^2\,\{C_3^{2\alpha_1}R^{2\alpha_1/(p-1)(1-\beta)}C_4R^{\{2+p(2\beta-1)\}/(p-1)}\\
&\qquad+(C_3R^{1/(p-1)(1-\beta)})^{2\alpha_2-1}\,C_2R^{2/(p-1)}\}\\
&\quad< R^{1/(p-1)(1-\beta)}.
\end{aligned}$$

Now we can repeat the routine work as in the last part of the proof of Proposition 4.4.

II. *The case* $r_0=R^{(2-p)/(p-1)}\geqslant T$. It suffices to repeat the same argument as above with K replaced by

$$\begin{aligned}
K:=\Big\{&v\in L^2(0,T;H);\int_0^T|v(t)|\,dt\leqslant RT,\int_0^T|v(t)|^2\,dt\\
&\leqslant R^{\{1+(p-2)(1-\beta)\}/(p-1)(1-\beta)}T\Big\}.
\end{aligned}$$

For example, with obvious modifications, one can obtain (4.41), (4.43),

$$\int_0^T\varphi^t(u(t))\,dt\leqslant 2C_1R^qT<C_2R^{2/(p-1)}$$

and

$$\int_0^T|g(t)|^2\,dt\leqslant C_4R^{2\beta q}T.$$

Q.E.D.

Proof of Remark 3.1. Let $h,f\in K:=\{v\in L^2(0,T;H);\ \int_0^T|v(t)|\,dt\leqslant R^{1/(p-1)},\ \int_0^T|v(t)|^2\,dt\leqslant R^{1/(p-1)(1-\beta)}\}$. Then, by much the same argument as before, it is easy to obtain (4.41), (4.43), $\int_0^T\varphi^t(u(t))\,dt\leqslant CR^{2/(p-1)}$ and $\int_0^T|g(t)|^2\,dt\leqslant CR^{4\beta/(p-1)}T^{1-2\beta}$. Hence, for each $b(t)\in B(t,u(t))$, we find

$$\begin{aligned}
\int_0^T|b(t)|\,dt\leqslant C(&R^{\alpha_1/(p-1)(1-\beta)}R^{2\beta/(p-1)}T^{1-\beta}\\
&+R^{(\alpha_2-1/2)/(p-1)(1-\beta)}R^{1/(p-1)}T^{1/2}),\\
\int_0^T|b(t)|^2\,dt\leqslant C(&R^{2\alpha_1/(p-1)(1-\beta)}R^{4\beta/(p-1)}T^{1-2\beta}\\
&+R^{(2\alpha_2-1)/(p-1)(1-\beta)}R^{2/(p-1)}).
\end{aligned}$$

Thus, for a sufficiently small $R>0$, $\mathbf{B}_{\pi,f,T,K}$ maps K into itself. Q.E.D.

Proof of Remark 3.2. Put $\phi_\varepsilon^t(u) = \varphi^t(u) + \varepsilon |u|_H^2/2$ and $B_\varepsilon(t, u) = B(t, u) - \varepsilon u$, $\varepsilon > 0$. Then $\partial\phi_\varepsilon^t$ becomes strictly monotone, since $\partial\phi_\varepsilon^t(u) = \partial\varphi^t(u) + \varepsilon u$ for all $u \in D(\partial\phi_\varepsilon^t) = D(\partial\varphi^t)$. Moreover ϕ_ε^t satisfies $(\mathrm{A}.\phi_\varepsilon^t)_{p,\beta}$ with (2.2) replaced by

$$\begin{aligned}\phi_\varepsilon^t(x(t)) \leqslant \phi_\varepsilon^{t_0}(x_0) &+ m_3 |t - t_0| (\phi_\varepsilon^{t_0}(x_0) + m_2)\\ &+ \varepsilon m_3 |t - t_0| (\varphi^{t_0}(x_0) + m_2)^{\beta + 1/p}/K_0^{1/p}\\ &+ \varepsilon m_3 |t - t_0|^2 (\varphi^{t_0}(x_0) + m_2)^{2\beta}/2. \end{aligned} \tag{2.2}'$$

Then, since ϕ_ε^t and $B_\varepsilon(t, \cdot)$ still satisfy (4.15), applying the same reasoning as in the proof of Proposition 4.4 for a sufficiently small $\varepsilon > 0$, one can solve the equation:

$$\left.\begin{aligned} &du(t)/dt + \partial\phi_\varepsilon^t(u(t)) + B_\varepsilon(t, u(t)) \ni f(t), \qquad 0 < t < T,\\ &u(0) = u(T), \end{aligned}\right| \quad (\mathrm{P.P.})'$$

which is equivalent to the original problem (P.P.).

As for the case of Theorem II, to solve (P.P.)$'$, it suffices to note that, from (4.41),

$$\sup_{t \in [r_0, T]} \int_{t - r_0}^{t} (|\varepsilon u(s)| + |\varepsilon u(s)|^2)\, ds \leqslant \varepsilon r_0 (C_1 R^{1/(p-1)} + \varepsilon C_1^2 R^{2/(p-1)})$$

can be taken sufficiently small. Q.E.D.

5. Application

In the section, we shall exemplify the applicability of our main results to periodic problems of some nonlinear heat equations and Navier–Stokes-type equations in bounded regions with periodically moving boundaries. Let T be a given positive number, and $Q(t)$ be a bounded domain in $\mathbf{R}_x^n$ with smooth boundary $\Gamma(t)$ for each $t \in [0, T]$. Put $Q(r, s) = \bigcup_{r<t<s} (Q(t) \times \{t\})$, $Q[0, T] = \bigcup_{0 \leqslant t \leqslant T} (Q(t) \times \{t\})$, $Q = Q(0, T)$ and $\Gamma = \bigcup_{0<t<T} (\Gamma(t) \times \{t\})$.

Throughout this section, we always assume the following conditions (Q.0), (Q.1) and (Q.3).

(Q.0) $Q(0) \equiv Q(T)$.

(Q.1) For each $t \in [0, T]$, $\Gamma(t)$ is sufficiently smooth (say, of C^3).

(Q.2) Q is covered by m slices $Q(s_i, t_i)$ $(i = 1, 2,..., m)$ such that for each $1 \leqslant i \leqslant m$, $Q(s_i, t_i)$ is mapped onto a cylindrical domain $Q(s_i) \times (s_i, t_i)$ by a diffeomorphism ψ_i which is of class C^3 up to the boundary and preserves the time coordinate t.

In stating our results, we shall use the following notations: Let $\mathscr{O}$ be an auxiliary open ball in $\mathbf{R}_x^n$ such that the closure of Q is contained in $\mathscr{O} \times [0, T]$. For each function v defined on $Q[0, T]$, we mean by $\hat{v}$ the zero extention of v to $\mathscr{O} \times [0, T]$, i.e., $\hat{v} = v$ in $Q[0, T]$ and $\hat{v} = 0$ in $\mathscr{O} \times [0, T] \backslash Q[0, T]$. Moreover, we denote by $C_\pi([0, T]; X(Q(t)))$ the set of all functions v defined on $Q[0, T]$ such that $v(\cdot, t)$ belongs to $X(Q(t))$ for all $t \in [0, T]$; $\hat{v}(\cdot, t)$ is an $X(\mathscr{O})$-valued continuous functions on $[0, T]$; and that $v(\cdot, 0) = v(\cdot, T)$, where $X(\Omega)$ ($\Omega = Q(t)$ or $\mathscr{O}$) denotes some function spaces defined on Ω such as $L^2(\Omega)$, $W_0^{1,p}(\Omega)$, etc.

EXAMPLE I. *Nonlinear heat equations.* We consider the following periodic problem for the nonlinear heat equations.

$$\left.\begin{array}{lll} \dfrac{\partial u}{\partial t}(x, t) = \Delta_p u + |u|^\alpha u + f(x, t) & \text{in} & Q, \\ u(x, t) = 0 & \text{on} & \Gamma, \\ u(x, 0) = u(x, T) & \text{in} & Q(0) = Q(T), \end{array}\right| \qquad (\mathrm{Pr.NH})_\pi$$

where Δ_p is the nonlinear Laplace operator, i.e.,

$$\Delta_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right), \qquad p \geqslant 2.$$

Our results are stated as follows:

THEOREM 5.1. (*The case* $2 + \alpha < p$.) *Let* $2 + \alpha < p$ *and the following* (α.1) *be satisfied.*

$$\left.\begin{array}{lll} -1 < \alpha < +\infty & \text{if} & n \leqslant p, \\ -1 < \alpha < np/2(n-p) - 1 & \text{if} & n > p. \end{array}\right| \quad (\alpha.1)$$

Let $f \in L^2(Q)$, *then* $(\mathrm{Pr.NH})_\pi$ *has a strong solution* u *satisfying*

$$u(\cdot, t) \in C_\pi([0, T]; W_0^{1,p}(Q(t))), \tag{5.1}$$

$$\partial u/\partial t, \qquad \Delta_p u \in L^2(Q). \tag{5.2}$$

THEOREM 5.2. (*The case* $2 + \alpha > p$). *Let* $2 + \alpha > p$ *and* (α.1) *be satisfied. Then there exists a* (*sufficiently small*) *positive number* r *independant of* T *such that if* $\sup_{1 \leqslant t \leqslant T}(\int_{t-1}^t |f(s)|_{L^2(Q(s))}^2 ds)^{1/2} \leqslant r$, *then* $(\mathrm{Pr.NH})_\pi$ *has a strong solution* u *satisfying* (5.1) *and* (5.2).

In the case of $p=2$, i.e., $\Delta_p=\Delta$, the conditions in the above theorem can be weakened as follows.

THEOREM 5.3. *Let $p=2$ and the following condition $(\alpha.2)$ on α be stisfied.*

$$\left.\begin{array}{lll} 0<\alpha<+\infty & \text{if} & n\leqslant 2, \\ 0<\alpha\leqslant 4/(n-2) & \text{if} & n>3. \end{array}\right| \quad (\alpha.2)$$

Then the assertion of Theorem 5.2 holds true with $p=2$.

To prove these theorems, we put

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\int_{\mathcal{O}} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^p dx \qquad \text{if} \quad u\in W_0^{1,p}(\mathcal{O}), \\ &= +\infty \qquad \text{if} \quad u\in L^2(\mathcal{O})\backslash W_0^{1,p}(\mathcal{O}), \end{aligned}$$

$$K(t)=\{u\in L^2(\mathcal{O});\, u(x)=0 \text{ for a.e. } x\in\mathcal{O}\backslash Q(t)\},$$

$$\begin{aligned} I_{k(t)}(u) &= 0 \qquad \text{if} \quad u\in K(t), \\ &= +\infty \qquad \text{if} \quad u\in L^2(\mathcal{O})\backslash K(t), \end{aligned}$$

$$\varphi^t(u)=\varphi(u)+I_{K(t)}(u) \qquad \text{for all} \quad u\in L^2(\mathcal{O}).$$

Then we have

$$\begin{aligned} D(\varphi^t) &= \{u\in L^2(\mathcal{O});\, u|_{Q(t)}\in W_0^{1,p}(Q(t)),\, u|_{\mathcal{O}\backslash Q(t)}=0\}, \\ D(\partial\varphi^t) &= \{u\in D(\varphi^t);\, -\Delta_p u|_{Q(t)}\in L^2(Q(t))\}, \\ \partial\varphi^t(u) &= \{f\in L^2(\mathcal{O}); f|_{Q(t)}=-\Delta_p u|_{Q(t)}\} \qquad \text{for} \quad u\in D(\partial\varphi^t). \end{aligned}$$

We define another operator $B(t,\cdot)$ by

$$\begin{aligned} B(t,u) &= -|u|^\alpha u \qquad \text{for} \quad u\in D(\partial\varphi^t), \\ D(B(t,\cdot)) &= D(\partial\varphi^t). \end{aligned}$$

Then, assumption $(\alpha.1)$ and Sobolev's theorem give,

$$|B(t,u)|_{L^2(\mathcal{O})}\leqslant C(\varphi^t(u))^{(1+\alpha)/p} \qquad \text{for all} \quad u\in D(\varphi^t), \tag{5.4}$$

$$(B(t,u),u)_{L^2(\mathcal{O})}\leqslant C(\varphi^t(u))^{(2+\alpha)/p} \qquad \text{for all} \quad u\in D(\varphi^t). \tag{5.5}$$

Thus $(\text{Pr.NH})_\pi$ is reduced to the following abstract periodic problem in $L^2(\mathcal{O})$:

$$\left.\begin{array}{ll} d\hat{u}(t)/dt+\partial\varphi^t(\hat{u}(t))+B(t,\hat{u}(t))\ni\hat{f}(t), & 0<t<T, \\ \hat{u}(0)=\hat{u}(T). & \end{array}\right| \quad (\text{P.P.})_1$$

Proof of Theorem 5.1. Let $t_0 \in [0, T]$ and $v_0 \in D(\varphi^{t_0})$. Since there exists a positive constant τ_0 such that $Q(I(t_0)) := Q(\max(0, t_0 - \tau_0), \min(t_0 + \tau_0, T))$ is containned in a slice $Q(s_i, t_i)$, using the representation $X^i(x, t)$ of ψ_i, we can define

$$v(x, t) = v_0((X^i)^{-1}(X^i(x, t), t_0)) \qquad \text{for} \quad x \in Q(t),$$
$$= 0 \qquad \text{for} \quad x \in \mathscr{O} \backslash Q(t),$$

for each $t \in I(t_0)$. Then, using (Q.1) and (Q.2), we can show that $v(\cdot, t) \in D(\varphi^t)$ for all $t \in I(t_0)$ and there exists a positive constant m such that

$$|v(\cdot, t) - v_0(\cdot)|_{L^2(\mathscr{O})} \leqslant m\,|t - t_0|\,(\varphi^{t_0}(v_0))^{1/p} \qquad \text{for all} \quad t \in I(t_0), \tag{5.6}$$
$$\varphi^t(v(t)) \leqslant \varphi^{t_0}(v_0) + m\,|t - t_0|\,\varphi^{t_0}(v_0) \qquad \text{for all} \quad t \in I(t_0). \tag{5.7}$$

By Poincaré's inequality, we also have $\varphi^t(u) \geqslant C\,|u|^p_{L^2(\mathscr{O})}$.

Thus $(\mathrm{A}.\varphi^t)_{p,1/p}$ is verified. Furthermore, (A.1) and (A.2) are easily derived from Rellich's compactness theorem and the demiclosedness of the operator $u \longmapsto |u|^\alpha u$. Since $\langle \partial\varphi^t(u), u\rangle_{L^2(\mathscr{O})} = p\varphi^t(u)$ for all $u \in D(\partial\varphi^t)$, (5.4) and (5.5) with $2 + \alpha < p$ assures (A.3). Hence we can apply Theorem I to $(\mathrm{P.P.})_1$ and the desired strong solution u is given by $u = \hat{u}|_Q$. Q.E.D.

Proof of Theorem 5.2. Since $2 + \alpha > p$ implies $(1 + \alpha)/p > (p - 1)/p$, $(\mathrm{A}.4)_{p,1/p}$ is assured by (5.4). Then we can apply Theorem II to $(\mathrm{P.P.})_1$. Q.E.D.

Proof of Theorem 5.3. Since $p = 2$, φ^t satisfies $(\mathrm{A}.\varphi^t)_{2,1/2}$ with $m_2 = 0$. Furthermore, using Sobolev's theorem and interpolation inequality, we deduce

$$|B(t, u)|_{L^2(\mathscr{O})} \leqslant C\,|\partial^0\varphi^t(u)|_{L^2(\mathscr{O})}\,(\varphi^t(u))^{\alpha/2} \qquad \text{for all} \quad u \in D(\partial\varphi^t),$$

which assures $(\mathrm{A}.4)_{2,1/2}$. Then Theorem II can be applied to $(\mathrm{P.P.})_1$. Q.E.D.

Remark. As for the critical case $p = 2 + \alpha$, we note:

(1) If the embedding constant C in (5.5) is strictly smaller than p, then (A.3) is satisfied, that is, Theorem 5.1 holds true with $p = 2 + \alpha$.

(2) Let $\alpha > 0$ and $(\alpha.1)$ be satisfied. Then, since $(1 + \alpha)/p = (1 + \alpha)/(2 + \alpha) > \frac{1}{2}$, (5.4) implies $(\mathrm{A}.4)_{p,\beta}$ with $\alpha_2 > \frac{1}{2}$. Therefore, as is mentioned in Remark 3.1, there exists a sufficiently small positive number r *depending on* T such that if $\sup\{\int_{t-1}^t |f(s)|^2\,ds;\ 1 \leqslant t \leqslant T\} \leqslant r$, then $(\mathrm{Pr.NH})_\pi$ has a strong solution.

EXAMPLE II. *Modefied Navier–Stokes equations.* For $\Omega = Q(t)$ or $\mathscr{O}$, we use the following notations:

$$\mathbf{H}(\Omega) = (L^2(\Omega))^n,$$

$$\mathbf{H}_\sigma(\Omega) = \text{the completion of } \mathbf{C}_\sigma^\infty(\Omega) \text{ under the } \mathbf{H}(\Omega)\text{-norm},$$

$$\mathbf{C}_\sigma^\infty(\Omega) = \{\mathbf{u} = (u^1, u^2, ..., u^n);\, u^i \in C_0^\infty(\Omega),\, i = 1, 2, ..., n,\, \operatorname{div} \mathbf{u} = 0\},$$

$$\mathbf{W}_\sigma^{1,p}(\Omega) = (W_\sigma^{1,p}(\Omega))^n \cap \mathbf{H}_\sigma(\Omega), \qquad \mathbf{H}_\sigma^1(\Omega) = \mathbf{W}_\sigma^{1,2}(\Omega),$$

$$P_\Omega = \text{the orthogonal projection from } \mathbf{H}(\Omega) \text{ onto } \mathbf{H}_\sigma(\Omega).$$

We consider the following periodic problems for the Navier–Stokes-type equations in the noncylindrical domain Q:

$$\left.\begin{array}{rll} \dfrac{\partial \mathbf{u}}{\partial t}(x, t) + A\mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \mathbf{f} - \nabla p^* & \text{in} & Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in} & Q, \\ \mathbf{u} = 0 & \text{on} & \Gamma, \\ \mathbf{u}(x, 0) = \mathbf{u}(x, T) & \text{in} & Q(0) = Q(T), \end{array}\right| \qquad (\text{Pr.MNS})_\pi$$

where the unknown $\mathbf{u}$ and given $\mathbf{f}$ are n-dimensional vector functions, and unknown p^* is a real scalar function. For the case that $A = -\Delta$ and $n = 3$, i.e., the 3-dimensional Navier–Stokes equation, this problem was already studied by Morimoto [12] in a class of weak solutions and by Ôtani–Yamada [17] in a class of strong solutions. Our main concern here is the existence of strong solutions for the following two types of equations (cf. Lions [11]):

$$\text{(I)} \qquad A\mathbf{u} = A_\alpha^1 \mathbf{u} = -(1 + \nu\, \|\mathbf{u}\|^\alpha)\, \Delta\mathbf{u}, \qquad \nu \geqslant 0, \quad \alpha \geqslant 0,$$

where

$$\|\mathbf{u}\| = |\nabla \mathbf{u}|_{L^2(Q(t))},$$

$$|\nabla \mathbf{u}|^2 = \sum_{i,j=1}^n \left| \frac{\partial u^i}{\partial x_j} \right|^2, \qquad \mathbf{u} = (u^1, u^2, ..., u^n).$$

(Note that if $\nu = 0$, then $(\text{Pr.MNS})_\pi$ coincides with the Navier–Stokes equation.)

$$\text{(II)} \qquad A\mathbf{u} = A_p^2 \mathbf{u} = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|\nabla \mathbf{u}|^{p-2} \frac{\partial \mathbf{u}}{\partial x_i} \right), \qquad p \geqslant 2.$$

In parallel with Theorems 5.1, 5.2 and 5.3 in [15], we have the following results.

THEOREM 5.4. *Let $A = A^1_\alpha$ with $\nu \geqslant 0$ and $\alpha \geqslant 0$ if $n = 2$, and $\nu > 0$ and $\alpha \geqslant 1$ if $n = 3$. Then, for all $\mathbf{f}(t) \in L^2(0, T; \mathbf{H}(Q(t)))$, $(\mathrm{Pr.MNS})_\pi$ has a strong solution $\mathbf{u}$ satisfying*

$$\mathbf{u}(\cdot, t) \in C_\pi([0, T]; \mathbf{H}^1_\sigma(Q(t))), \tag{5.8}$$

$$\partial \mathbf{u}(\cdot, t)/\partial t, \qquad \Delta \mathbf{u}(\cdot, t) \in L^2(0, T; \mathbf{H}(Q(t))). \tag{5.9}$$

THEOREM 5.5. *Let $A = A^1_\alpha$ with $\nu \geqslant 0$ and $\alpha \geqslant 0$, and let $n = 3$ or 4. Then there exists a (sufficiently small) positive number r independant of T such that if $\sup_{1 \leqslant t \leqslant T} (\int_{t-1}^t |\mathbf{f}(s)|^2_{\mathbf{H}(Q(s))}\, ds)^{1/2} \leqslant r$, then $(\mathrm{Pr.MNS})_\pi$ has a strong solution $\mathbf{u}$ satisfying* (5.8) *and* (5.9).

THEOREM 5.6. *Let $A = A^2_p$ with $p > 2$ and $p \geqslant 4n/(n+2)$. Then, for all $\mathbf{f}(t) \in L^2(0, T; \mathbf{H}(Q(t)))$, $(\mathrm{Pr.MNS})_\pi$ has a strong solution $\mathbf{u}$ satisfying*

$$\mathbf{u}(\cdot, t) \in C_\pi([0, T]; \mathbf{W}^{1,p}_\sigma(Q(t))), \tag{5.10}$$

$$\partial \mathbf{u}(\cdot, t)/\partial t \in L^2(0, T; \mathbf{H}(Q(t))). \tag{5.11}$$

Proof of Theorems 5.4 *and* 5.5. We employ much the same procedure as in the proof of Theorem 5.1 in [15]. Put

$$\begin{aligned}
\varphi(\mathbf{u}) &= \tfrac{1}{2} \int_{\mathscr{O}} |\nabla \mathbf{u}|^2\, dx \qquad &&\text{if} \quad \mathbf{u} \in \mathbf{H}^1_\sigma(\mathscr{O}),\\
&= +\infty &&\text{if} \quad \mathbf{u} \in \mathbf{H}_\sigma(\mathscr{O}) \backslash \mathbf{H}^1_\sigma(\mathscr{O}),\\
K(t) &= \{\mathbf{u} \in \mathbf{H}_\sigma(\mathscr{O});\, \mathbf{u}(x) = 0 \text{ for } x \in \mathscr{O} \backslash Q(t)\}, && \qquad\qquad (5.12)\\
I_{K(t)} &= 0 \qquad \text{if} \quad \mathbf{u} \in K(t)\\
&= +\infty \qquad \text{if} \quad \mathbf{u} \in H_\sigma(\mathscr{O}) \backslash K(t),\\
\varphi^t(\mathbf{u}) &= \varphi(\mathbf{u}) + \frac{\nu}{2+\alpha} (2\varphi(\mathbf{u}))^{(2+\alpha)/2} + I_{K(t)}(\mathbf{u}) \qquad &&\text{for all} \quad \mathbf{u} \in \mathbf{H}_\sigma(\mathscr{O}).
\end{aligned} \tag{5.13}$$

Then we have

$$\begin{aligned}
D(\varphi^t) &= \{\mathbf{u} \in H_\sigma(\mathscr{O});\, \mathbf{u}|_{Q(t)} \in \mathbf{H}^1_\sigma(Q(t)),\, \mathbf{u}|_{\mathscr{O} \backslash Q(t)} = 0\},\\
D(\partial \varphi^t) &= \{\mathbf{u} \in D(\varphi^t);\, \mathbf{u}|_{Q(t)} \in (H^2(Q(t)))^n\},\\
\partial \varphi^t(\mathbf{u}) &= \{\mathbf{f} \in \mathbf{H}_\sigma(\mathscr{O});\, P_{Q(t)} \mathbf{f}|_{Q(t)} = -P_{Q(t)} A^1_\alpha u|_{Q(t)}\}.
\end{aligned}$$

We further define $B(t, \cdot)$ by

$$\left|\begin{aligned}
&B(t, \mathbf{u}) = P_{\mathscr{O}}(\mathbf{u} \cdot \nabla)\, \mathbf{u} \qquad \text{for} \quad \mathbf{u} \in D(\partial \varphi^t),\\
&D(B(t, \cdot)) = D(\partial \varphi^t).
\end{aligned}\right. \tag{5.14}$$

Then $(\text{Pr.MNS})_\pi$ with $A = A_\alpha^1$ is reduced to the following abstract periodic problem $(\text{P.P.})_2$ in $\mathbf{H}_\sigma(\mathscr{O})$:

$$(\text{P.P.})_2 \quad \left| \begin{array}{ll} d\hat{u}/dt + \partial\varphi^t(\hat{\mathbf{u}}(t)) + B(t, \hat{\mathbf{u}}(t)) \ni P_{\mathscr{O}}\hat{\mathbf{f}}(t), & 0 < t < T, \\ \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}(T), & \end{array} \right.$$

where $\hat{\mathbf{f}}(t)$ is the zero extension of $\mathbf{f}\ (t)$ to $\mathscr{O}$.

Since $(B(t, \mathbf{u}), \mathbf{u})_{\mathbf{H}_\sigma(\mathscr{O})} = 0$, $\langle \partial\varphi^t(\mathbf{u}), \mathbf{u} \rangle_{\mathbf{H}_\sigma(\mathscr{O})} \geqslant 2\varphi^t(\mathbf{u})$ for all $\mathbf{u} \in D(\partial\varphi^t)$ and $\varphi^t(\mathbf{u}) \geqslant C\,|\mathbf{u}|^2_{\mathbf{H}_\sigma(\mathscr{O})}$ for all $\mathbf{u} \in D(\varphi^t)$, conditions (A.1) and (ii) of (A.3) are fulfilled. Moreover, by the standard argument, conditions (A.2) and $(\text{A}.\varphi^t)_{2,1/2}$ with $m_2 = 0$ are verified. (See, e.g., [15] and [17].)

As for the boundedness condition on $B(t, \cdot)$, we have (see, e.g., Ladyzhenskaya [10], Fujita and Kato [6] and Temam [18]), if $n = 2$, then

$$|B(t, \mathbf{u})|^2_{\mathbf{H}_\sigma(\mathscr{O})} \leqslant C\,|\mathbf{u}|_{\mathbf{H}_\sigma(\mathscr{O})}(\varphi^t(\mathbf{u}))\,|\partial^0\varphi^t(\mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})}, \tag{5.15}$$

if $n = 3$, then

$$|B(t, \mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})} \leqslant (C/\nu)(\varphi^t(\mathbf{u}))^{3/(4+2\alpha)}\,|\partial^0\varphi^t|^{1/2}_{\mathbf{H}_\sigma(\mathscr{O})}, \qquad \nu > 0 \tag{5.16}$$

or

$$|B(t, \mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})} \leqslant C(\varphi^t(\mathbf{u}))^{3/4}\,|\partial^0\varphi^t(\mathbf{u})|^{1/2}_{\mathbf{H}_\sigma(\mathscr{O})}, \tag{5.17}$$

and if $n = 4$, then

$$|B(t, \mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})} \leqslant C(\varphi^t(\mathbf{u}))^{1/2}\,|\partial^0\varphi^t(\mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})}. \tag{5.18}$$

Hence (5.15) or (5.16) with $\nu > 0$ and $\alpha \geqslant 1$ assures (i) of (A.3), and (5.17) or (5.18) assures (ii) of $(\text{A}.4)_{2,1/2}$. Then Theorems I and II can be applied for $(\text{P.P.})_2$ and the desired solution $\mathbf{u}$ is given by $\mathbf{u} = \hat{\mathbf{u}}|_Q$. Q.E.D.

Proof of Theorem 5.6. We put $\varphi^t(\mathbf{u}) = \varphi(\mathbf{u}) + I_{K(t)}(\mathbf{u})$, where $\varphi(\mathbf{u}) = |\nabla\mathbf{u}|^p_{L^p(\mathscr{O})}/P$ if $\mathbf{u} \in \mathbf{W}_\sigma^{1,p}(\mathscr{O})$, $= +\infty$ if $\mathbf{u} \in \mathbf{H}_\sigma(\mathscr{O}) \backslash \mathbf{W}_\sigma^{1,p}(\mathscr{O})$, and $I_{K(t)}$ is the indicator function of $K(t)$ defined by (5.12) and (5.13). We also define $B(t, \cdot)$ by (5.14). Then $(\text{Pr.MNS})_\pi$ with $A = A_p^2$ is again reduced to $(\text{P.P.})_2$ in $\mathbf{H}_\sigma(\mathscr{O})$. Furthermore we see that $(\text{A}.\varphi^t)_{p,1/p}$ and the following estimate hold,

$$|B(t, \mathbf{u})|_{\mathbf{H}_\sigma(\mathscr{O})} \leqslant C(\varphi^t(\mathbf{u}))^{2/p} \qquad \text{for all} \quad \mathbf{u} \in D(\varphi^t),$$

since $\mathbf{W}_\sigma^{1,p}(\mathscr{O})$ is embedded in $(L^{2p/(p-2)}(\mathscr{O}))^n$. Then we can apply Theorem I to $(\text{P.P.})_2$. Q.E.D.

APPENDIX

Proof of Lemma 4.5. Since $\varphi^t(\cdot) \geqslant 0$, it is clear that $D(\varphi^t) = D(\varphi^t_\varepsilon)$ and $D(\varphi^t_{\varepsilon,r}) = D(\varphi^t) \cap \{u \in H; |u|_H \leqslant r\}$.

In order to verify (i), it suffices to replace the function $x(t)$ given in (2.1) and (2.2) by $\tilde{x}(t) = x(t)$ if $|x(t)| \leqslant r$, and $\tilde{x}(t) = rx(t)/|x(t)|$ if $|x(t)| \geqslant r$. Indeed, let $x_0 \in D(\varphi^{t_0}_{\varepsilon,r})$, then in the case of $|x(t)| > r$, we have

$$\begin{aligned}
|\tilde{x}(t) - x_0| &\leqslant \frac{r}{|x(t)|}|x(t) - x_0| + \frac{|x(t)| - r}{|x(t)|}|x_0| \\
&\leqslant |x(t) - x_0| + |x(t)| - |x_0| \\
&\leqslant |x(t) - x_0| \\
&\leqslant 2m_1 |t - t_0| (\varphi^{t_0}(x_0) + m_2)^\beta \\
&\leqslant 2m_1 |t - t_0| ((3\varphi^{t_0}_{\varepsilon,r}(x_0)/\varepsilon)^{1/3} + m_2)^\beta
\end{aligned} \tag{a.1}$$

and

$$\begin{aligned}
\varphi^t_{\varepsilon,r}(\tilde{x}(t)) &= \frac{r}{|x(t)|}\varphi^t_\varepsilon(x(t)) + \frac{(|x(t)| - r)}{|x(t)|} \cdot \varphi^t_\varepsilon(0) \\
&\leqslant \varphi^t(x(t)) + \varepsilon(\varphi^t(x(t)))^3/3 + (K_1 + \varepsilon K_1^3/3)\,|x(t) - x_0|/r \\
&\leqslant \varphi^{t_0}_{\varepsilon,r}(x_0) + m_3' |t - t_0| (\varphi^{t_0}_{\varepsilon,r}(x_0) + m_2'),
\end{aligned} \tag{a.2}$$

where m_2' and m_3' are constants depending on m_2, m_3, δ_0, r and K_1 but not on ε.

The second assertion is a direct consequence of Corollary 2.11 of Brezis [4], since $D(\varphi^t_\varepsilon) \cap \operatorname{Int} D(I_r)$ contains $\{0\}$. As for the last assertion, it is clear that $D(\partial\varphi^t) \subset D(\partial\varphi^t_\varepsilon)$ and $(1 + \varepsilon(\varphi^t(u))^2)\,\partial\varphi^t(u) \subset \partial\varphi^t_\varepsilon(u)$ for all $u \in D(\partial\varphi^t)$, since

$$\begin{aligned}
&\varepsilon(\varphi^t(v))^3/3 - \varepsilon(\varphi^t(u))^3/3 \\
&\qquad \geqslant \varepsilon(\varphi^t(u))^2\,(\varphi^t(v) - \varphi^t(u)) \geqslant (\varepsilon(\varphi^t(u))^2\,g, v - u)
\end{aligned}$$

for all $v \in H$, $u \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(u)$.

Therefore, in order to verify (iii), it suffices to show that $(1 + \varepsilon(\varphi^t(\cdot))^2)\,\partial\varphi^t(\cdot)$ is maximal monotone.

To this end, for an arbitrary $f \in H$, let us consider the equation,

$$u_\lambda + (1 + \lambda)\,\partial\varphi^t(u_\lambda) \ni f, \qquad \lambda \geqslant 0, \tag{a.3}$$

and the operator $\Phi = \Phi(f)$, $\Phi : \lambda \mapsto \varepsilon(\varphi^t(u_\lambda))^2$.

Then, by the standard argument, it is easy to obtain the following a priori estimates.

$$\tfrac{1}{2}|u_\lambda|^2 + (1+\lambda)\,\varphi^t(u_\lambda) \leqslant (1+\lambda)\,\varphi^t(v) + 2(|v|^2 + |f|^2)$$
$$\text{for all} \quad v \in D(\varphi^t), \tag{a.4}$$

$$|u_\lambda - u_\mu|^2 \leqslant (\lambda - \mu)(\varphi^t(u_\mu) - \varphi^t(u_\lambda)), \tag{a.5}$$

$$|\varphi^t(u_\lambda) - \varphi^t(u_\mu)| \leqslant (|g_\lambda| + |g_\mu|)\,|u_\lambda - u_\mu|,$$

where $g_\lambda = (f - u_\lambda)/(1+\lambda) \in \partial\varphi^t(u_\lambda)$.

Hence it follows that Φ is continuous in λ. On the other hand, since $\varphi^t(u_\lambda)$ is monotone decreasing in λ by (a.5), we find by (a.4) that there exists a positive number L such that

$$\max_{0 \leqslant \lambda \leqslant L} \Phi(\lambda) = \varepsilon(\varphi^t(u_0))^2$$
$$\leqslant \varepsilon\{\varphi^t(v) + 2(|v|^2 + |f|^2)\}^2 \leqslant L.$$

Thus Φ is a continuous mapping which maps $[0, L]$ into itself. Then, by Schander's fixed point theorem, there exists a $\lambda \in [0, L]$ such that $\lambda = \Phi(\lambda)$, i.e., $u_\lambda + (1 + \varepsilon(\varphi^t(u_\lambda))^2)\,\partial\varphi^t(u_\lambda) \ni f$. Q.E.D.

References

1. H. Attouch and A. Damlamian, On multivalued evolution equations in Hilbert spaces, *Israel J. Math.* **12** (1972), 373–390.
2. P. Bénilan and H. Brézis, Solutions faibles d'équations d'évolution dans les espaces de Hilbert, *Ann. Inst. Fourier (Grenoble)* **22** (1972), 311–329.
3. M. Biroli, Sur une équation d'évolution multivoque et non monotone dans un espace de Hilbert, *Rend. Sem. Mat. Univ. Padova* **52** (1974), 313–331.
4. H. Brézis, "Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert," Math. Studies, Vol. 5, North-Holland, Amsterdam/New York, 1973.
5. H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, *in* "Proceedings, Symposium in Pure Mathematics," Vol. 18, pp. 105–113, Amer. Math. Soc., Providence, R.I., 1970.
6. H. Fujita and T. Kato, On the Navier–Stokes initial value problem, I, *Arch. Rational Mech. Anal.* **16** (1970), 269–315.
7. H. Ishii, Asymptotic stability and blowing up of solutions of some nonlinear equations, *J. Differential Equations* **26** (1977), 291–319.
8. N. Kenmochi, Some nonlinear parabolic variational inequalities, *Israel J. Math.* **22** (1975), 304–331.
9. Y. Koi and J. Watanabe, On nonlinear evolution equations with a difference term of subdifferentials, *Proc. Japan Acad.* **52** (1976), 413–416.
10. O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," Gordon & Breach, New York, 1969.

11. J. L. LIONS, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod, Paris, 1969.
12. H. MORIMOTO, On existence of periodic weak solutions of the Navier–Stokes equations in regions with periodically moving boundaries, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **18** (1971/1972), 499–524.
13. T. NAGAI, Periodic solutions for certain time-dependent parabolic variational inequalities. *Hiroshima Math. J.* **5** (1975), 537–549.
14. ÔTANI, On existence of strong solutions for $du(t)/dt + \partial\psi^1(u(t)) - \partial\psi^2(u(t)) \ni f(t)$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **24** (1977), 575–605.
15. M. ÔTANI, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, *J. Differential Equations* **46** (1982), 268–299.
16. M. ÔTANI, Existence and asymptotic stability of strong solutions of nonlinear evolutation equations with a difference term of subdifferentials, *in* "Qualitative Theory of Differential Equations," Colloquia Math. Soc. János Bolyai No. 30, North-Holland, Amsterdam, 1981.
17. M. ÔTANI AND Y. YAMADA, On the Navier–Stokes equations in non-cylindrical domains: An approach by the subdifferential operator thery, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **25** (1978), 185–204.
18. R. TEMAM, "Navier–Stokes Equations," Studies in Mathematics and its Application, Vol. 2, North-Holland, New York, 1977.
19. M. TSUTSUMI, Existence and nonexistence of global solutions for nonlinear parabolic equations, *Publ. RIMS Kyoto Univ.* **8** (1972), 211–229.
20. Y. YAMADA, Periodic solutions of certain nonlinear parabolic differential equations in domains with periodically moving boundaries, *Nagoya Math. J.* **70** (1980), 111–123.